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Abstract

The method of separation of variables for partial differential equations is formulated in terms of direct products of Hilbert spaces. By this formulation we can determine the possible boundary conditions for which the method is applicable. Finally, a contour integral representation is given for the solution of some partial differential equations.

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1. Introduction

The method of separation of variables is one of the few methods by which partial differential equations can be solved explicitly. The method, however, is obviously limited to those cases where the partial differential equation expressed in a suitable coordinate system, has solutions which are products of functions of the separate coordinates; consequently, much research has been devoted to the determination of the coordinate systems in which particular equations, such as the wave equation, have separable solutions. Along with this, there has been a study of the special functions defined by the separate differential equations. However, even when the separable coordinate systems and the special functions needed have been determined, there is still one question to be discussed, namely the question of the type of boundary conditions for which the method of separation of variables can be used. To answer this question, it was found convenient to place it and the whole method of separation of variables in an abstract framework.

Using the concept of the direct product of Hilbert spaces*, we show in Theorem I that the idea of separation of variables leads to a method for solving operator equations in which the operator is a sum of direct products of operators in the factor spaces. The formulation of Theorem I has many advantages. First, it enables us to discuss how boundary conditions affect the use of separation of variables. For example, we show that certain boundary-value problems for hyperbolic equations have a well-defined solution. Also, some problems which do not seem to be amenable to the method of separation of variables can be solved with the aid of Theorem I. Finally, this theorem suggests a formal method of solving partial differential equations with a minimum of calculation.

* The concept has been used by Cordes [6] in studying the question of the completeness of the products of the separated solutions. However, the ideas and methods of his paper differ completely from those of the present paper.

In Theorem II we consider a different aspect of the method of separation of variables. In many problems which are solved by this method there exist alternative representations for the solution. For example, in solving Poisson's equation for a square with zero boundary conditions, the solution may be expressed either as a sum of sines of multiples of x or as a sum of sines of multiples of y. Each representation may be obtained from the original problem by separating the variables in a different order. In Theorem II we obtain a contour-integral representation of the solutuion which by suitable deformation of the contour gives either one representation or the other. Theorem II also shows that the Green's function for a partial differential equation may be written as the convolution of the Green's function for the separated ordinary differential equations. Finally, a study of the form of the solution given in Theorem II sheds light on the role played by the Sommerfeld radiation condition in ensuring the uniqueness of the solution of boundary-values problems for infinite domains.

2. The abstract formulation

Let $\mathcal{H}_1$ be a separable Hilbert space containing elements $a, a_1, a_2, \ldots,$ $u, u_1, u_2, \ldots,$ f and let $A_1, A_2, \ldots, A_k$ be self-adjoint operators on $\mathcal{H}_1$. Let $\mathcal{H}_2$ be another separable Hilbert space containing elements $b, b_1, b_2, \ldots, v, v_1, v_2 \ldots g$ and let $B_1, B_2, \ldots, B_k$ be self-adjoint operators on $\mathcal{H}_2$. We form the product space of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the manner described by Murray and von Neumann[1]. Denote this product space by $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the elements in it by w, h, $a \otimes v$, $u \otimes b, \ldots,$ where, as stated before, a and u belong to $\mathcal{H}_1$, and v and b belong to $\mathcal{H}_2$.

The scalar products in the spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ will be denoted by the symbols

$$\langle u, f \rangle_1, \quad \langle v, g \rangle_2, \quad \langle w, h \rangle_{12}$$

respectively. The scalar product in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is related to the scalar products in $\mathcal{H}_1$ and $\mathcal{H}_2$ as follows: If

$$w = \sum_1^m u_i \otimes v_i,$$

$$h = \sum_1^n f_j \otimes g_j,$$

then

(1) $$\langle w,h \rangle_{12} = \sum_{i=1}^m \sum_{j=1}^n \langle u_i, f_j \rangle_1 \cdot \langle v_i, g_j \rangle_2 .$$

Note that the scalar product with the subscript 1 is defined only for two elements in $\mathcal{H}_1$; however, if w is the element of $\mathcal{H}_1 \otimes \mathcal{H}_2$ defined above and if u is any element of $\mathcal{H}_1$, we put

(2) $$\langle v,w \rangle_2 = \sum_{i=1}^m \langle u, u_i \rangle_1 \, v_i .$$

Similarly, if v is any element of $\mathcal{H}_2$, we put

(3) $$\langle v,w \rangle_2 = \sum_{j=1}^m \langle v, v_i \rangle_2 \, u_i .$$

It is known[1] that if $u_1, u_2, \ldots$ form a complete orthonormal set in $\mathcal{H}_1$, then any element h in $\mathcal{H}_1 \otimes \mathcal{H}_2$ may be represented uniquely as follows:

(4) $$h = \sum_1^\infty u_n \otimes b_n,$$

where

(5) $$b_n = \langle u_n, h \rangle_1 .$$

Also, the Parseval equality holds, i.e.,

$$\langle h,h\rangle_{12} = \sum_{1}^{\infty} \langle b_n, b_n\rangle_2. \tag{6}$$

Let A be a self-adjoint operator acting on a domain $\mathcal{D}_1$ contained in $\mathcal{H}_1$ and let B be a self-adjoint operator acting on a domain $\mathcal{D}_2$ contained in $\mathcal{H}_2$. We define the operator $A \otimes B$ as follows: If $u_1, \dots, u_m$ are elements in $\mathcal{D}_1$ and if $v_1, \dots, v_m$ are elements in $\mathcal{D}_2$, then

$$A \otimes B \sum_{1}^{m} u_j \otimes v_j = \sum_{1}^{m} A\, u_j \otimes B\, v_j.$$

We call $A \otimes B$ the direct product of the operators A and B.

To illustrate these concepts, we may take $\mathcal{H}_1$ as the space of complex-valued functions $f(x)$, $0 \leq x \leq 1$, such that $\int_0^1 |f(x)|^2\, dx < \infty$, and take $\mathcal{H}_2$ as the space of complex-valued functions $g(y)$, $0 \leq y \leq 1$, such that $\int_0^1 |g(y)|^2 dy < \infty$; then $\mathcal{H}_1 \otimes \mathcal{H}_2$ will be the space of complex-valued functions $h(x,y)$, $0 \leq x,\ y \leq 1$, such that

$$\int_0^1 \int_0^1 |h(x,y)|^2\, dx\, dy < \infty .$$

The scalar products will be defined by the following formulas:

$$\langle f_1, f_2\rangle_1 = \int_0^1 \overline{f_1(x)}\, f_2(x)\, dx,$$

$$\langle g_1, g_2\rangle_2 = \int_0^1 \overline{g_1(x)}\, g_2(x)\, dx,$$

and

$$\langle h_1, h_2\rangle_{12} = \int_0^1 \overline{h_1(x,y)}\, h_2(x,y) dx\, dy.$$

Suppose that

$$A\, u = - \frac{d^2}{dy^2}\, u$$

for all functions u(x) in $\mathcal{H}_1$ such that u(x) has a square-integral second derivative and such that u(0) = u(1) = 0. Suppose also that

$$B\,v = -\frac{d^2}{dy^2}\,v$$

for all functions v(y) in $\mathcal{H}_2$ such that v(y) has a square-integrable second derivative and such that v(0) = v(1) = 0. Let I_1 and I_2 denote the identity operator in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively; then, for all functions w(x,y) in $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that w(x,y) has square-integrable second partial derivatives with respect to x and y, and such that w(0,y) = w(1,y) = w(x,0) = w(x,1) = 0, we have

$$\left(A \otimes I_2 + I_1 \otimes B\right)\; w(x,y) = -\frac{\partial^2 w}{dx^2} - \frac{\partial^2 w}{\partial y^2} = -\Delta\, w.$$

We shall now prove a theorem which provides an abstract formulation of the method of separation of variables for solving partial differential equations. Consider a self-adjoint operator L in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and suppose that

$$L = A_1 \otimes B_1 + \dots + A_k \otimes B_k, \tag{7}$$

where $A_1, A_2, \dots, A_k$ are self-adjoint operators in $\mathcal{H}_2$. We say that the operators $A_1, \dots, A_k$ have a <u>common spectral representation</u> if there exists a non-decreasing family of projections $E(\lambda)$, $-\infty < \lambda < \infty$, and a set of real-valued functions $\alpha_1(\lambda), \dots, \alpha_k(\lambda)$ such that

$$E(-\infty) = 0,\; E(\infty) = 1,$$

and such that for any element u in the common domain of $A_1, \dots, A_k$ we have

$$A_j u = \int_{-\infty}^{\infty} \alpha_j(\lambda)\; d\,E(\lambda)u, \qquad 1 \le j \le k. \tag{8}$$

If the spectrum of each of the operators A_j is discrete, the concept of a common spectral representation has the following meaning: There exists a complete orthonormal set of elements $u_1, u_2, \ldots$ in $\mathcal{H}_1$ and a set of real numbers α_{jn}, $1 \leq j \leq k$, $1 \leq n < \infty$ such that

(9) $$A_j\, u_n = \alpha_{jn}\, u_n, \qquad 1 \leq j \leq k,\ 1 \leq n \leq \infty.$$

The numbers α_{jn} are the eigenvalues and the elements u_n are the eigenelements of A_j. Thus in this case a common spectral representation implies the existence of a common set of eigenelements.

We now prove the following:

Theorem I. Suppose that L, the operator defined in (7), is a closed operator in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and suppose that the operators $A_1, \ldots, A_k$ have a common spectral representation, i.e., suppose (8) holds. Then the equation

$$L\, w = h,$$

where h is any element in $\mathcal{H}_1 \otimes \mathcal{H}_2$, will have a solution if the operators

(10) $$\left(\alpha_1(\lambda)B_1 + \ldots + \alpha_k(\lambda)B_k\right)^{-1}$$

in $\mathcal{H}_2$ are uniformly bounded for all λ such that $d\,E(\lambda) > 0$. When (9) holds, condition (10) may be replaced by the requirement that the operators

(11) $$\left(\alpha_{1n}B_1 + \ldots + a_{kn}\, B_k\right)^{-1}$$

be uniformly bounded for all n. If (8) holds, the solutuion is given by the formula

(12) $$w = \int d\, E(\lambda) \otimes \left(\alpha_1(\lambda)B_1 + \ldots + \alpha_k(\lambda)B_k\right)^{-1} h$$

and if (9) holds, the solution is

(13) $$w = \sum_n u_n \otimes \left(\alpha_{1n}\, B_1 + \ldots + \alpha_{kn}B_k\right)^{-1} b_n$$

where b_n is defined by (5).

Since the proof for (12) involves only a slight modification of the proof for (13), we shall prove (13) only. Consider the sum

$$w_{mp} = \sum_{n=m}^{p} u_n \otimes \left(a_{1n}B_1 + \dots + a_{kn}B_k\right)^{-1} b_n.$$

Because the u_n are orthonormal, we have

$$\langle w_{mp}, w_{mp}\rangle_{12} = \sum_n \left\langle \left(a_{1n}B_1 + \dots + a_{kn}B_k\right)^{-1} b_n, \left(a_{1n}B_1 + \dots + a_{kn}B_k\right)^{-1} b_n\right\rangle_2$$

$$\leq M^2 \sum_{n=m}^{p} \langle b_n, b_n\rangle_2$$

if M is the uniform bound for the operators in (11). From (6) we see that

$$\sum_1^\infty \langle b_n, b_n\rangle_2 < \infty .$$

Therefore

$$\sum_{n=m}^{p} \langle b_n, b_n\rangle_2$$

and also w_{mp} converges to zero. This implies that

$$w_m = \sum_1^m u_n \otimes \left(a_{1n}B_1 + \dots + a_{kn}B_k\right)^{-1} b_n$$

approaches a limit as $m \to \infty$. Call this limit w.

It must still be proved that $Lw = h$. We have

$$A_1 \otimes B_1 w_m = \sum_1^m A_1 u_n \otimes B_1 \left(a_{1n}B_1 + \dots + a_{kn}B_k\right)^{-1} b_n$$

$$= \sum_1^m u_n \otimes a_{1n}B_1\left(a_{1n}B_1 + \dots + a_{kn}B_k\right)^{-1} b_n .$$

Therefore

$$L\, w_m = \sum_1^m u_n \otimes \left(a_{1n}B_1 + \dots + a_{kn}B_k\right) \left(a_{1n}B_1 + \dots a_{kn}B_k\right)^{-1} b_n$$

$$= \sum_1^m u_n \otimes b_n.$$

Since the elements u_n form a complete set in $\mathcal{H}_1$, the sum $\sum_1^m u_n \otimes b_n$ converges to h as m approaches infinity. By hypothesis L is a closed operator. Since w_m converges to w and $L w_m$ converges to h, this implies that $L w = h$. This completes the proof of our theorem.

It is well known[2] that if the operators $A_1, \ldots, A_k$ commute with each other, they have a common spectral representation. Using this result, we may obtain a generalization of Theorem I.

Let $D_1, \ldots, D_k$ be operators in $\mathcal{H}_1$ such that D^{-1} has a bounded inverse and such that the operators $D_1^{-1} D_2, \ldots, D_1^{-1} D_k$ are self-adjoint, commute, and have discrete spectra. Then if we put $A_2 = D_1^{-1} D_2, \ldots, A_k = D_1^{-1} D_k$, we see that there exists a complete set of orthonormal vectors $u_1, u_2, \ldots$ such that

$$A_j u_n = \alpha_{jn} u_n, \qquad 2 \leq j \leq k, \quad 1 \leq n < \infty,$$

and such that

(14) $$D_j u_n = \alpha_{jn} D_1 u_n, \qquad 2 \leq j \leq k.$$

Using (14) and the methods of Theorem I, we prove the following:

<u>Corollary</u>. Let $D_1, \ldots, D_k$ be the operators described above and let $B_1, \ldots, B_k$ be self-adjoint operators in $\mathcal{H}_2$. Suppose that

$$M = D_1 \otimes B_1 + \ldots + D_k \otimes B_k$$

is a closed operator in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Suppose that condition (11) is satisfied with α_{jn} defined by (14). Then the equation

$$M w' = h',$$

where h' is any element in $\mathcal{H}_1 \otimes \mathcal{H}_2$, has the solution

$$w' = \sum u_n \otimes \left(\alpha_{1n} B_1 + \ldots + \alpha_{kn} B_k\right)^{-1} b_n',$$

where

$$b_n' = \langle D_1 u_n, h' \rangle_1 .$$

3. Applications

We have said that Theorem I is an abstract formulation of the method of separation of variables. To see this, suppose that

$$L = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \qquad 0 \leq x, \quad y \leq 1$$

on the domain of complex-valued functions $w(x,y)$ such that their second partial derivatives $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{dy^2}$ are square integrable and such that

$$w(0,y) = w(1,y) = w(x,0) = w(x,1) = 0 .$$

Then, as we have seen before, we may put

$$L = A \otimes I_2 + I_1 \otimes B,$$

where $A = -\frac{d^2}{dx^2}$, $B = -\frac{d^2}{dy^2}$, and where I_1 and I_2 are the identity operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, the spaces of square integrable functions of x and y, respectively. Since A and I_1 commute, they have a common spectral representation. In fact, the functions $u_n = \sqrt{2}\ \sin n\pi x$, $n = 1,2,\ldots$ form a complete orthonormal set of $\mathcal{H}_1$, and

$$I_1 u_n = u_n,$$

$$A\ u_n = n^2 \pi^2 u_n.$$

From Theorem I the solution of the equation $Lw = h$ is

$$w = 2 \sum_1^{\infty} (n^2 \pi^2 + B)^{-1} b_n(y) \sin n\pi x, \tag{15}$$

where

$$b_n(y) = \int_0^1 h(x,y) \sin n\pi x\, dx.$$

Put

$$(n^2 \pi^2 + B)^{-1} b_n(y) = c_n(y).$$

Then

$$(16) \qquad -\frac{d^2 c_n}{dy^2} + n^2 \pi^2 c_n = b_n$$

and $c_n(0) = c_n(1) = 0$. The solution of the equation

$$-\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = h$$

with the boundary conditions

$$w(0,y) = w(1,y) = w(x,0) = w(x,1) = 0$$

is found by the method of separation of variables to be

$$w = 2 \sum c_n(y) \sin n\pi x,$$

where c_n satisfies (16). Since this solution is the same as (15), we see that Theorem I contains the method of separation of variables.

Theorem I can also be used, however, to solve certain problems which cannot be solved by separation of variables. For example, consider the following problem:

Find a function $w(x,y)$ such that

$$(17) \qquad L\,w = \frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = h, \qquad 0 \le x \le 1, \quad 0 \le y < \infty,$$

and such that

$$w(x,0) = w_y(x,0) = 0\,, \quad w(0,y) = w(1,y) = 0$$

and either

a) $w_x(0,y) = w_x(1,y) = 0$

or

b) $w_{xx}(0,y) = w_{xx}(1,y) = 0.$

Eq. (17) cannot be solved by the method of separation of variables because, if we assume $X(x)Y(y)$ is a solution of the homogeneous equation, we get

$$\frac{X^{(IV)}}{X} + 2\,\frac{X''}{X}\,\frac{Y''}{Y} + \frac{Y^{(IV)}}{Y} = 0,$$

and clearly in this equation the function of x is not separable from the function of y. However, (17) can be solved by the use of Theorem I if the boundary conditions b) are assumed. To see this, put $A_1 = \frac{d^4}{dx^4}$ and $A_2 = -\frac{d^2}{dx^2}$ with the boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0$$

and put $B_2 = -\frac{d^2}{dy^2}$, $B_3 = \frac{d^4}{dy^4}$, $0 \leq y < \infty$ with the boundary conditions $v(0) = v'(0) = 0$. Then

$$L = A_1 \otimes I_2 + 2\,A_2 \otimes B_2 + I_1 \otimes B_3.$$

The operators A_1, A_2 and I_1 have a common spectral representation defined by the complete set of eigenfunctions $u_n = \sqrt{2}\,\sin n\pi x$, $n = 1, 2, \ldots$. Since

$$A_2 u_n = n^2\,\pi^2\,u_n$$

and

$$A_1 u_n = n^4\,\pi^4\,u_n,$$

we need only show, in order to be able to apply Theorem I, that the operators

$$\left(B_3 + 2\,n^2\pi^2 B_2 + n^4\pi^4\right)^{-1}$$

are uniformly bounded. The proof of the existence of this bound follows easily from the fact that the operators B_2 and B_3 are positive-definite. Using (13) we may obtain an explicit representation of the solution for (17). The result is not necessary for our purposes; the interest of this problem is that it cannot be solved by separation of variables but can be solved by Theorem I.

Note, however, that if we assume boundary conditions a), Theorem I cannot be used. We may define B_2 and B_3 as before, but now $A_1 = \frac{d^4}{dx^4}$ and $A_2 = \frac{d^2}{dx^2}$, with the boundary conditions

$$u(0) = u(1) = u'(0) = u(1) = 0.$$

These operators A_1 and A_2 do not have a common spectral representation and therefore our methods do not apply. In fact, an explicit representation of the solution of (17) with the boundary conditions a) is not known.

Another important consequence of Theorem I is that it provides a proof of the existence of a solution to the equation

$$L\,w = h.$$

However, this existence is proved only if a condition such as (10) or (11) is satisfied. It is interesting that this condition may be satisfied even in cases where the general theory of partial differential equations seems to state that no solution is possible. For example, according to the general theory, a boundary-value problem for a hyperbolic equation is an unreasonable problem. However, consider the following:

Find a function $w(x,y)$ satisfying the hyperbolic equation

$$L\,w = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = h(x,y) \tag{18}$$

and satisfying the boundary conditions

$$w(0,y) = w(1,y) = w(x,0) = w_y(x,1) = 0.$$

If we put

$$L = A \otimes I_2 + I_1 \otimes B,$$

where $A = -\frac{d^2}{dx^2}$, with the boundary conditions $u(0) = u(1) = 0$, and $B = +\frac{d^2}{dy^2}$ with the boundary conditions $v(0) = v'(1) = 0$, then we find that the eigenvalues of A are $n^2\pi^2$ and the operators $(B + n^2\pi^2)^{-1}$ are uniformly bounded. Consequently,

Theorem I may be applied and Eq. (18) has a solution. However, if we change the boundary condition $w_y(x,1) = 0$ to $w_y(x,1 + \varepsilon) = 0$, then there exist arbitrarily small values of ε for which Eq. (17) will not have a solution. This type of behavior, in which an infinitesimal variation in a parameter, such as ε, may cause a problem to change from one having a solution to one not having a solution and then back again, is physically unreasonable because no parameter can ever be measured exactly. This is the basis for the theoretical conclusion that the boundary-value problem for a hyperbolic problem is unreasonable.

Theorem I has an important practical consequence. The form of the solution for w shows that the operators $\alpha_{1n}B_1 + \dots + \alpha_{kn}B_k$ must be inverted. This means that we may consider the operators $A_1, \dots, A_k$ as constants, invert the operator $A_1B_1 + \dots + A_k B_k$ (here $A_1,\dots, A_k$ represent constants), and then interpret the result by using the spectral representation for $A_1,\dots,A_k$.

An illustration will clarify this method. Consider the problem of finding a function $w(x,y)$ such that

$$(19) \qquad -\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = h(x,y) \qquad 0 \leq x \leq 1, \ -\infty < y < \infty$$

and such that

$$w(0,y) = w(1,y) = 0.$$

Put $A = -\dfrac{d^2}{dx^2}$, with the boundary conditions $u(0) = U(1) = 0$. Then (19) may be written as

$$\frac{d^2 w}{dy^2} - A\,w = -\,h(x,y)\ .$$

The solution of this equation (assuming w is not exponentially large at $y = \pm\,\infty$) is

$$w = (4A)^{-1/2} \int_{-\infty}^{\infty} \exp\left[-A^{1/2}|y-\eta|\right] h(x, \eta)\, d\eta.$$

In order to interpret this result we may use the operational calculus of Dunford[3] or, what is equivalent, the spectral representation of A. The eigenfunctions of A are $u_n = \sqrt{2}$ sin $n\pi x$ and the corresponding are $n^2\pi^2$. Since

$$f(A)u_n = f(n^2\pi^2)\; u_n,$$

and since

$$h(x, \eta) = 2 \sum_1^\infty \sin n\pi x \int_0^1 \sin n\pi x\, \xi\; h(\xi, \eta) d\xi,$$

we see that

$$w = \sum_1^\infty \frac{\sin n\pi x}{n\pi} \int_0^1 \sin n\pi\xi\; d\xi \int_{-\infty}^{\infty} e^{-n\pi|y-\eta|} h(\xi, \eta) d\eta .$$

Thus the desired solution to (19) has been obtained with a minimum of calculation.

4. The Euler-Poisson-Darboux equation

The methods of the preceding section can be applied to the discussion of the Euler-Poisson-Darboux equation[4], namely,

$$\text{(20)} \qquad \triangle w = w_{tt} + k\, t^{-1}\, w_t,$$

with the initial conditions $w = f$ and $w_t = 0$ for $t = 0$. Here the Laplacian is an operator in m space variables. For convenience, put $\triangle = -p^2$, and consider Eq. (20) with p a constant. It becomes

$$\text{(21)} \qquad w_{tt} + k\, t^{-1}\, w_t + p^2 w = 0.$$

The solution of this equation is

$$w = a_1 t^\alpha J_\alpha(pt) + a_2 t^\alpha J_{-\alpha}(pt),$$

where

$$\alpha = (1 - k)/2$$

and where a_1 and a_2 are independent of t, and J_α and $J_{-\alpha}$ are Bessel functions. To satisfy the initial conditions we must find a solution of (21) such that $w(0) = f$ and $w_t(0) = 0$. From the power series expansion for the Bessel functions we have

$$J_\alpha(pt) = \frac{1}{\Gamma(\alpha+1)} \left(\frac{pt}{2}\right)^\alpha \left[1 + O(t^2)\right].$$

Consequently, for $\alpha > \frac{1}{2}$ the function $t^\alpha J_\alpha(pt)$ and its first derivative both vanish for $t = 0$. For $\alpha < \frac{1}{2}$, the derivative of the function $t^\alpha J_\alpha(pt)$ becomes infinite for $t = 0$. On the other hand, the value of the function $t^\alpha J_{-\alpha}(pt)$ at $t = 0$ is $\frac{1}{\Gamma(1-\alpha)} \left(\frac{2}{p}\right)^\alpha$ while its derivative vanishes at $t = 0$. Using these results, we see that the solution of (21) which satisfies the initial conditions is

$$\text{(22)} \qquad w = \Gamma\left(\frac{1+k}{2}\right) \left(\frac{pt}{2}\right)^\alpha J_{-\alpha}(pt)f.$$

To interpret this we must use the spectral representation for $\Delta = -p^2$. The spectral representation will be obtained by using the Fourier transform in m dimensions. Let $\underline{x}$, $\underline{\xi}$ denote vectors in m-dimensional Euclidean space and $\underline{p}$ a vector of m dimensions in the transformed space. Then we have

$$f(x) = (2\pi)^{-m} \int \cdots \int e^{i\underline{p}\cdot(\underline{x}-\underline{\xi})} f(\underline{\xi})d\underline{p}\, d\underline{\xi},$$

where the integration is extended over the entire p and ξ space. Also, if $\phi(-\Delta)$ is an analytic function of Δ, we have

$$\phi(-\Delta)f(x) = (2\pi)^{-m} \int \cdots \int \phi(p^2) e^{i\underline{p}\cdot(\underline{x}-\underline{\xi})} f(\underline{\xi})d\underline{\xi}\, d\underline{p},$$

where $p^2 = \underline{p} \cdot \underline{p}$. Using this, we see that (22) becomes

$$\text{(23)} \qquad w = C_k \int \cdots \int (pt)^\alpha J_{-\alpha}(pt)\, e^{i\underline{p}\cdot(\underline{x}-\underline{\xi})}\, f(\underline{\xi})d\underline{\xi}\, d\underline{p},$$

where

$$C_k = \Gamma\left(\frac{1+k}{2}\right) 2^{-\alpha}(2\pi)^{-m}.$$

By interchanging the order of integration, (23) can be reduced to the formulas given elsewhere[4]. Similar methods have been proposed in an abstract by P.C. Rosenblum[5].

The above discussion has shown that the Euler-Poisson-Darboux equation has a unique solution for $k \geq 0$ but not for $k < 0$. However, for $k < 0$ we may consider the solution (23) which is the analytic continuation of the solution for $k \geq 0$. As can be seen from the formula for C_k, this solution breaks down when $k = -1, -3, -5, \ldots$. This breakdown could have been foreseen because the exceptional values of k correspond to integral values of α and for these values of α the functions J_α and $J_{-\alpha}$ are not linearly independent.

Let us consider the exceptional values of k. They occur when $k = 1-2n$ or $\alpha = -n$, with n a positive integer. For these values of k we take the following solution of (21):

$$w = -\frac{\pi}{(n-1)!}\left(\frac{pt}{2}\right)^n N_n(pt)f, \tag{24}$$

where $N_n(pt)$ is the Neumann function. This solution has the disadvantage that its 2n-th derivative becomes logarithmically infinite for $t = 0$. However, for certain kinds of functions $f(\underline{x})$, the infinity will not occur; consequently, w and all of its derivatives will be continuous at $t = 0$. To find these functions, note that

$$(pt)^n N_n(pt) = \sum_0^\infty C_j (pt)^{2j} + (pt)^{2n} \log(pt) \sum_0^\infty C_j' (pt)^{2j},$$

where C_j and C_j' are scalars. The solution w will not have any logarithmic term if

$$(pt)^{2n} f(\underline{x}) = 0,$$

that is, if

$$\Delta^n f(\underline{x}) = 0. \tag{25}$$

Thus, we have obtained Weinstein's result [4] that if f(x) satisfies (25), then there exists a solution of (20) which is continuous and has continuous derivatives of all orders.

5. Alternative representations

Formula (15) is that solution of the equation

$$-\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = h \tag{26}$$

which satisfies zero boundary conditions on the sides of the square $x = 0$, $x = 1$, $y = 0$, $y = 1$. This solution is expressed in terms of the x-operator eigenfunctions $\sin n\pi x$. Since the problem is symmetric, it is to be expected that there will exist a representation of the solution in terms of the y-operator eigenfunctions, namely $\sin n\pi y$. This expectation is correct, and the desired representation can be found by using Theorem I, since the operators $-\frac{d^2}{dy^2}$ commute with I_2.

It would be desirable to obtain both representations without having to return to the original problem. This will be achieved by Theorem II, in which we shall obtain a contour integral representation for the solution. Deforming the contour one way will give a representation in terms of x-eigenfunctions; deforming the contour in another way will give a representation in terms of y-eigenfunctions.

Theorem II. Suppose that the closed operator L in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by the formula

$$L = A \otimes I_2 + I_1 \otimes B,$$

where A and B are self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Suppose that the distance between the spectrum of $-B$ and the spectrum of A is greater than zero. Then the solution of

$$L\, w = h,$$

where h belongs to $\mathcal{H}_1 \otimes \mathcal{H}_2$, is

$$(27) \qquad w = \frac{1}{2\pi i} \int_C (A - \lambda)^{-1} \otimes (B + \lambda)^{-1} \, d\lambda \, h,$$

where the region bounded by C contains the complete spectrum of A and is at a non-zero distance δ from the spectrum of $-B$.

As in Theorem I, we shall restrict the proof to the case where the spectrum of A is discrete and contains the points $\lambda_1, \lambda_2, \dots$. Let C_p be any closed contour containing the points $\lambda_1, \lambda_2, \dots, \lambda_p$ and let the domain bounded by C_p be at a distance greater then δ from the spectrum of $-B$. Consider the integral

$$w_p = (2\pi i)^{-1} \int_{C_p} (A - \lambda)^{-1} \otimes (B + \lambda)^{-1} \, d\lambda \, h.$$

From (4) we have

$$h = \sum u_n \otimes b_n,$$

and therefore

$$w_p = (2\pi i)^{-1} \int_{C_p} (\lambda_n - \lambda)^{-1} u_n \otimes (B + \lambda)^{-1} b_n \, d\lambda.$$

By the definition of C_p, the operator $B + \lambda$ is bounded by δ. It is easy to show that we may interchange integration and summation; then we obtain

$$w_p = \sum_1^p u_n \otimes (B + \lambda_n)^{-1} b_n.$$

But we have shown in the proof of Theorem I that w_p converges to a limit w which is the solution of

$$L w = h.$$

This proves Theorem II.

In the proof of Theorem II we have shown how (28) leads to an expansion in terms of A-eigenfunctions. Similarly, if the operator $(A-\lambda)^{-1} \otimes (B + \lambda)^{-1}$

is sufficiently small for large $|\lambda|$, we may deform the contour C to enclose the spectrum of $-B$ and thus obtain an expansion in terms of B-eigenfunctions. As an illustration, consider again (26). Here $A = -\frac{d^2}{dx^2}$, with the boundary conditions $u(0) = u(1) = 0$, and $B = -\frac{d^2}{dy^2}$, with the boundary conditions $v(0) = v(1) = 0$. The operators $(A-\lambda)^{-1}$ and $(B+\lambda)^{-1}$ are integral operators with kernels $g(x,\xi,\lambda)$ and $g(y, \quad ,-\lambda)$ respectively, where

$$g(x,\xi,\lambda) = \frac{\sin\sqrt{\lambda}\, x \sin\sqrt{\lambda}\,(1-\xi)}{\sqrt{\lambda}\sin\sqrt{\lambda}}, \qquad x < \xi,$$

$$= \frac{\sin\sqrt{\lambda}\,\xi \sin\sqrt{\lambda}\,(1-x)}{\sqrt{\lambda}\sin\sqrt{\lambda}}, \qquad x > \xi.$$

From Theorem II we have

(28) $$w = \frac{1}{2\pi i}\int_C d\lambda \int_0^1\int_0^1 g(x,\xi,\lambda)g(y,\eta,-\lambda)h(\xi,\eta)\,d\xi\,d\eta,$$

where C is the contour indicated in Fig. 1.

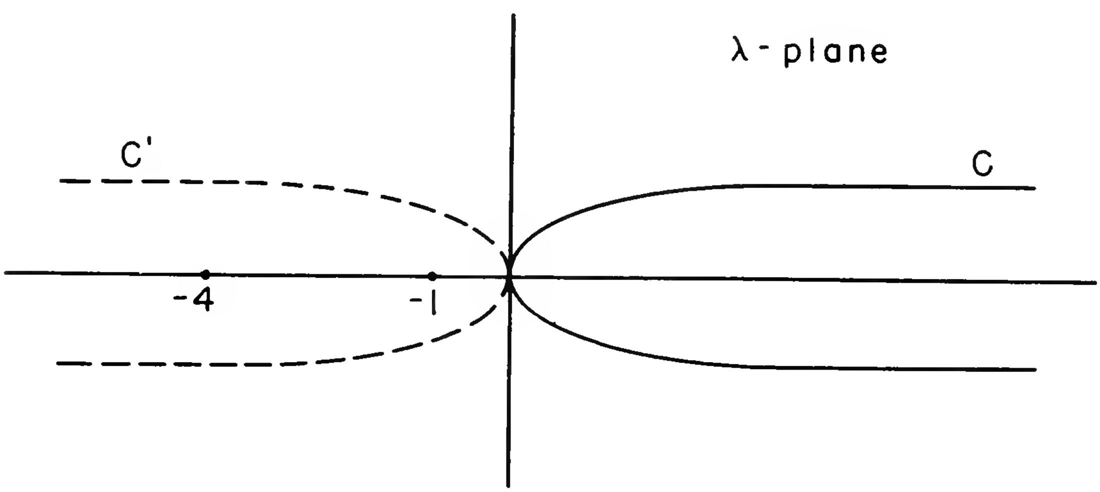

Figure 1

If we evaluate the residues inside C, we get (15). However, if we deform C into C', there will be poles at $\lambda = -1, -4, \ldots$ and by evaluating the residues at these poles we would get a representation for w in terms of $\sin n\pi y$.

Theorem II may also be written as follows:

$$\left(A \otimes I_2 + I_1 \otimes B\right)^{-1} = \frac{1}{2\pi i} \int_C (A-\lambda)^{-1} \otimes (B+\lambda)^{-1}\, d\lambda.$$

For differential operators the inverse operator is an integral operator. Let $G(\mu)$ be the kernel of the integral operator which inverts $L - \mu$ and let $g_1(\lambda)$, $g_2(\lambda)$ be the kernels of the integral operators which invert $A-\lambda$ and $B-\lambda$ respectively. Then Theorem II shows that

$$G(\mu) = (2\pi i)^{-1} \int_C g_1(\lambda) g_2(\mu - \lambda)\, d\lambda.$$

If we put $h = \delta\,(\xi - \xi')\delta(\eta - \eta')$ in (28), it will be an illustration of this result.

The following Corollary to Theorem II may be proved by a method similar to that used to prove the Corollary of Theorem I.

<u>Corollary</u>. Suppose that

$$L = A_1 \otimes B_1 + A_2 \otimes B_2,$$

where A_1, A_2 are operators in $\mathcal{H}_1$, and B_1, B_2 are operators in $\mathcal{H}_2$. Suppose that A_1 and B_2 have bounded inverses, that $A_1^{-1} A_2$ and $B_2^{-1} B_1$ are self-adjoint, and that the distance from the spectrum of $A_1^{-1} A_2$ to the spectrum of $B_2^{-1} B_1$ is positive. Then the solution of

$$L\, w = h$$

is

$$w = (2\pi i)^{-1} \int_C (A_2 - \lambda A_1)^{-1} \otimes (B_1 + \lambda B_2)^{-1}\, d\lambda\; h.$$

6. Interlacing spectra

In this section we shall discuss the possible relationships between the spectra of A and -B which permit the use of Theorem II. The simplest relationship is that illustrated by the problem (26), in which the spectrum of A is a set of discrete points on the positive real axis and the spectrum of -B is a set of discrete points on the negative real axis. This relationship will occur if A and B are both positive-definite operators.

If A is positive-definite and B is negative-definite, the spectrum of both A and -B will be on the positive real axis. This occurs in (18), where the spectrum of A consists of the points $\lambda_n = (nx)^2$ and the spectrum of -B consists of the points $\mu_n = (n + \frac{1}{2})^2\pi^2$. Since the distance from the set of λ_n to the set of μ_n is greater than zero, Theorem II may be applied and will give a result equivalent to that obtained previously. An appropriate contour C is illustrated in Fig. 2.

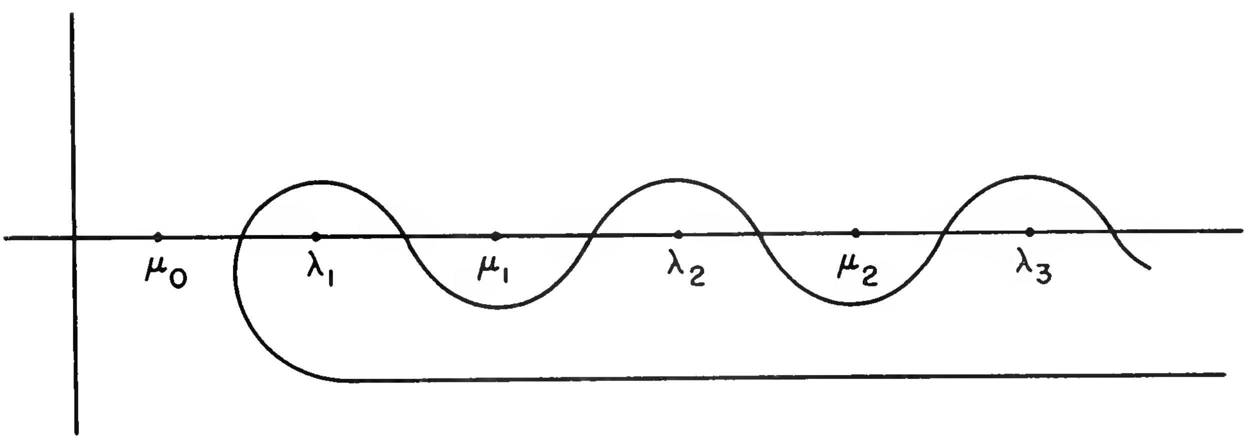

Figure 2

Let us see what happens to the spectrum of B if we change the length of the vetical side to $1 + \varepsilon$ so that the boundary conditions are $v(0) = v'(1+\varepsilon) = 0$. We find that the spectrum of -B consists of the points $\mu_m' = \frac{(2m+1)^2\pi^2}{4(1+\varepsilon)}$. Note that for arbitrary small values of ε there exist values of n and m such that $|\lambda_n - \mu_m'|$ can be made arbitrarily small. This shows that the spectra are so closely interlaced that we cannot find a contour C and thus we cannot apply Theorem II. In fact, for these values of ε the problem (18) does not have a solution.

If A is positive-definite and B is only bounded below, the spectrum of -B may partly overlap the spectrum of A. Suppose that the spectrum of A is discrete, with its only limit point at infinity. If the spectrum of -B is also discrete, with its only limit point at $-\infty$, then there are two possibilities: either the spectra have a positive distance between them, or there is a value of λ which is both in the spectrum of A and in the spectrum of -B. If first possibility occurs, Theorem II may be applied. If the second possibility occurs, the equation

$$L\, w = 0$$

will have a non-trivial solution, namely $w = u_n \otimes v_n$, where u_n, v_n are the eigenelements of the operators A and -B respectively, corresponding to the common eigenvalue λ_n. The question of the existence of a solution to $L\, w = h$ is now more difficult and will not be discussed.

If the spectrum of -B is continuous, it may overlap a point in the spectrum of A as is shown in Fig. 3 (the shaded part of the real axis represents the continuous spectrum of -B).

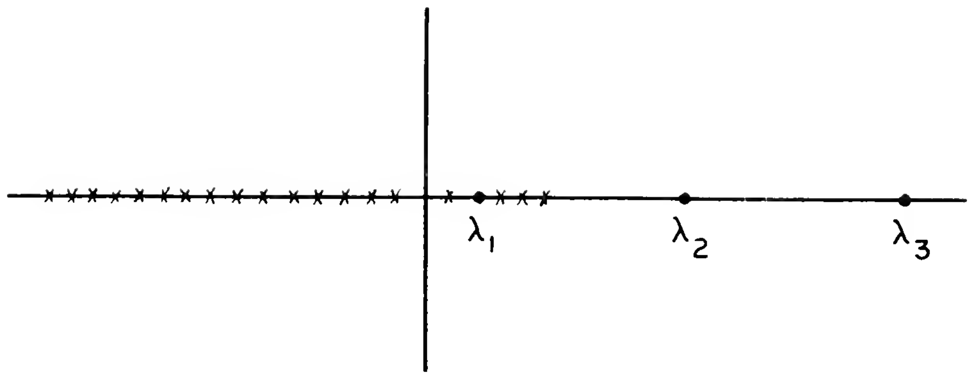

Figure 3

Since λ_1 is in the spectrum of $-B$, there is difficulty in obtaining a suitable contour C. If we consider an example in which this occurs, such as (19), we find that because of this overlap the problem is not well posed unless we add a condition at infinity, namely the Sommerfeld radiation condition. We may say that the role of this condition is to determine the behavior of the curve C in the neighborhood of λ_1 (see Fig. 4).

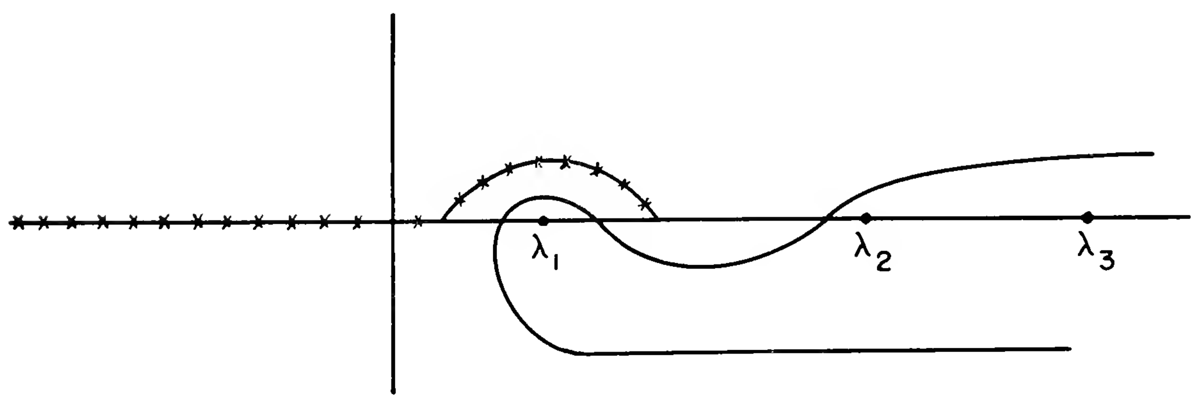

Figure 4

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The method of separation of variables for partial differential equations is formulated in terms of direct products of Hilbert spaces. By this formulation we can determine the possible boundary conditions for which the method is applicable. Finally, a contour integral representation is given for the solution of some partial differential equations.